1. Commutative Algebra and Homological Methods

Methods of Homological Algebra are used to characterize modules and rings with good properties in the sense that the notions of maximum number of regular elements and minimum number of parameters coincide. In this section we follow the presentation in [6]. Also for a development of the theory of sheaves we refer to [3] as well as [4] for the Algebraic Geometry.

1.1. **Depth and Cohen-Macaulay rings.** Let X be a topological space, $\mathcal{F}$ an abelian sheaf on X and Z a locally closed part of X. Let's choose a decomposition $Z = X \cap U$, where $U \subset X$ is open and let's put

$$\Gamma_Z(\mathcal{F}) = \{s \in \Gamma(U, \mathcal{F}) : \operatorname{supp}(s) \subset Z\}.$$

The definition does not depend on the choice of the open U, because if $U' \subset U$ is another open set, the map $\varphi : \Gamma_Z(\mathcal{F}) \to \Gamma'_Z(\mathcal{F})$ induced by the restriction map $\Gamma(U, \mathcal{F}) \to \Gamma(U', \mathcal{F})$ is an isomorphism.
The functor $\Gamma_Z(.)$ is left exact and the category of sheafs has enough injective so we can construct derived functors.

Definition 1.1. *One denotes $H^i_Z(F)$ the i-th right derived functor of Γ_Z and we called it the i-th local cohomology group with support in Z. For a quasi-coherent sheaf $\mathcal{F} = \tilde{M}$ on an affine scheme* $\operatorname{Spec}(A)$ *and a closed subset $Z = V(J)$, we denote $H^i_Z(\tilde{M})$ simply by $H^i_J(M)$.*

Definition 1.2. *Let A be a ring, M an A-module, J an ideal of A. One calls depth of M in J, denoted* $\operatorname{depth}_J(M)$, *the smallest integer i, such that $H^i_J(M) \neq 0$. For a local ring A with maximal ideal m,* $\operatorname{depth}_m(M)$ *is simply denoted* $\operatorname{depth}(M)$.

The notion of depth of a module M over a ring A is very related to the idea of M-regular elements in A.

Definition 1.3. *Let M be a A-module. A sequence $f_1, \ldots, f_n$ of non-invertible elements of A is said to be M-regular if for all i, f_i is not a zero divisor in $M/(f_1, \ldots, f_{i-1})M$.*

Proposition 1.4. *Let A be a local ring, m the maximal ideal and M an A-module of finite type. The following conditions are equivalents:*

1. $\operatorname{depth}(M) \geq n$.
2. *There exist an M-regular sequence $f_1, \ldots, f_n$ of elements in m.*

Proof. This is included in Proposition 1 in Chapter III of [6]. We will like to proceed by induction in n. The statement for $n = 1$ says that:

$$H^0_m(M) \neq 0 \Leftrightarrow \text{There are not regular elements in } m.$$

We can characterize $H^0_m(M) = \{s \in M : \exists n \geq 1, m^n s = 0\}$. The absence of regular elements in m forces $m \subset \cup_{p \in \mathrm{Ass}(M)} p$. Because M is of finite type, m has to be inside a finite sum: $m \subset \cup_{i=1}^n p_i$ and using the fact that the p_i are primes and the maximality of m we obtain $m = p_i = \mathrm{Ann}(s)$ for some $s \neq 0$. We have then $s \in H^0_m(M) \neq 0$. On the other hand if $s \in H^0_m(M) \neq 0$ we have $m^n \subset \mathrm{Ann}(s)$ and because m is maximal, $m = \mathrm{Ann}(s')$ for some $s' \in M$ and elements of m will not be regular.
The induction step: Consider a regular element $f \in m$ and the exact sequence coming from the multiplication map $f : M \to M$,

$$M \to M \to M/fM \to 0.$$

Applying $H^i_m(*)$ and denoting $N = M/fM$, we get the exact sequence

$$0 \to \cdots \to H^i_m(M) \to H^i_m(M) \to H^i_m(N) \to H^{i+1}_m(M) \to \dots$$

So if (1) $\Rightarrow$ (2) is true for n we will get a short exact sequence:

$$H^n_m(M) \to H^n_m(M) \to H^n_m(N) \to H^{n+1}_m(M)$$

and n elements N-regular $f_2, \dots, f_{n+1}$ in m. The elements $f, f_2, \dots, f_{n+1}$ will be $n+1$ M-regular elements in m and we will have that (1) $\Rightarrow$ (2) is true for $n+1$. In the other direction if (2) $\Rightarrow$ (1) is true for n and we have a sequence of M-regular elements $f_1, \dots, f_{n+1}$, the fact that $f_2, \dots, f_{n+1}$ is N-regular forces $H^{n-1}_m(N) = 0$ and we have the exact sequence:

$$0 \to H^n_m(M) \to H^n_m(M),$$

where the map $f : H^n_m(M) \to H^n_m(M)$ is the multiplication by f. This map can not be injective for $f \in m$ and therefore $H^n_m(M) = 0$. □

The depth and the dimension of a module M over a local ring A are related by the following inequality:

Proposition 1.5. *Let A be a noetherian local ring with maximal ideal m and M an A-module. Then* $\mathrm{depth}(M) \leq \dim(M)$.

Proof. This is Proposition 3 in Chapter III of [6]. The fact that for any $a \in A$ we have $\dim(M/aM) \geq \dim(M) - 1$ is a consequence of the Hilbert-Samuel theorem. Regular elements $f \in A$ on the other hand, are parameters in the sense that $\dim(M/fM) = \dim(M) - 1$. To see this, we take a regular element $f \in A$ and a chain of prime divisors $p_1 \subset p_2 \subset \cdots \subset p_d$ in $\mathrm{Supp}(M/fM)$ defining the dimension d of M/fM. Because f is regular it does not belong to the annihilator of any $s \in M$, in other words, it does not belong to any of the associated primes in $\mathrm{Ass}(M)$. In particular f is not part of any minimal prime in

Supp(M). But $f \in p_1$ as chosen before, so p_1 can not be minimal in M and there will be a prime p_0 in Supp(M) with the property

$$p_0 \subset p_1 \subset p_2 \subset \cdots \subset p_d$$

showing that $\dim(M/fM) \leq \dim(M) - 1$. □

Definition 1.6. *Let A be a local noetherian ring and M an A-module of finite type. M is said to be Cohen-Macaulay if* $\operatorname{depth}(M) = \dim(M)$*. A module M of finite type over a (non-necessarily local) ring A is called Cohen-Macaulay if the localizations M_m are all Cohen-Macaulay for maximal ideals m of A. A ring is Cohen-Macaulay if it is as a module over itself.*

1.2. **Equidimensional rings.** The geometric idea of equidimensional rings is an algebraic form of the notion of varieties all of whose irreducible components have the same dimension.

Definition 1.7. *A ring A is called equidimensional if it satisfies the two conditions:*

(1) $\dim(A) = \dim(A_m)$ *for every maximal ideal m of A and,*
(2) $\dim(A) = \dim(A/p)$ *for every minimal prime p of A.*

Remark 1.8. *A local ring that is Cohen-Macaulay must be equidimensional since it satisfies the relation*

$$\operatorname{height}(p) + \dim(A/p) = \dim(A),$$

for all prime ideals $p \subset A$. The geometric intuition is that the local ring of a point on an algebraic variety is not Cohen Macaulay if it is in the intersection of two irreducible components of different dimensions.

2. Algebraic Methods and Combinatorics: The Face Rings

Face rings are rings associated to the combinatorics of simplicial complexes and with applications to the geometry of toric varieties.

Definition 2.1. *Δ is called an abstract (finite) simplicial complex if $\Delta \subset 2^V$ where $|V| < \infty$ and $F \subset T \in \Delta \Rightarrow F \in \Delta$.*

Definition 2.2. *Let Δ be a $(d-1)$-dimensional simplicial complex on $Ver(\Delta) = [n]$. Let k be a field. Define $S = k[x_1, \dots, x_n]$ and define the face ideal I in S by $I = (\prod_{i \in J} x_i : J \subset [n], J \notin \Delta)$. Then define the face ring of Δ over k as $k[\Delta] := S/I_\Delta$.*

We note that in general, the ideal I_Δ is generated by the minimal non-faces of Δ . So

$$I_\Delta = (\prod_{i \in J} x_i : J \text{ is a minimal non face of } \Delta).$$

Definition 2.3. *A simplicial complex Δ is said Cohen-Macaulay (respectively Equidimensional, Unique factorization domain, Normal) when the associated face ring has that property.*

Example 2.4. *(The face ring of the triangle) Let A be the ring $A = k[v_1, v_2, v_3]/(v_1v_2v_3)$. The element $\lambda = v_1v_2v_3 \neq 0$ is regular in the Cohen-Macaulay domain $k[v_1, v_2, v_3]$. As a consequence the ring $A = k[v_1, v_2, v_3]/(v_1v_2v_3)$ is also Cohen-Macaulay with $\operatorname{depth}_m(A) = 2$ and $\dim(A) = \dim(A_m) = 2$ for any maximal ideal m.*

Example 2.5. *(The ring of the rectangle) In a similar way the ring $A = k[v_1, v_2, v_3, v_4]/(v_1v_3, v_2v_4)$ is Cohen-Macaulay with $\dim(A_m) = 2$ and $\operatorname{depth}_m(A) = 2$ for any maximal ideal $m \subset A$, because the elements $\lambda_1 = v_1v_3, \lambda_2 = v_2v_4$ form a regular sequence:*

$$P(v_1, v_2, v_3, v_4)\lambda_1 = Q(v_1, v_2, v_3, v_4)\lambda_2 \Rightarrow \lambda_2 \mid P(v_1, v_2, v_3, v_4).$$

Example 2.6. *The ring $A = k[v_1, v_2, v_3]/(v_1v_2, v_2v_3)$ is not equidimensional at $m = (v_1, v_2, v_3)$. The associated variety is the union of two irreducible components: $\Sigma_1 = V(v_2)$ with $\dim(\Sigma_1) = 2$ and $\Sigma_2 = V(v_1, v_3)$ with $\dim(\Sigma_2) = 1$. We have the following:*

1. *For points $x \in \Sigma_1 \setminus \Sigma_2$, $\dim(A_x) = \operatorname{depth}_{m_x}(A_x) = 2$.*
2. *For $x \in \Sigma_2 \setminus \Sigma_1$, $\dim(A_x) = \operatorname{depth}_{m_x}(A_x) = 1$.*
3. *For $x \in \Sigma_2 \cap \Sigma_1$, $\dim(A_x) = 2$ while $\operatorname{depth}_{m_x}(A_x) \leq 1$.*

In this case we can exhibit a system of parameters $x_1 = v_1 - v_3$ and $x_2 = v_2 - v_3$ in A. Indeed, $A/(x_1, x_2) = k[v_3]/(v_3^2)$ is of dimension zero, which shows that A is two-dimensional, as it is its maximal component Σ_1.
The sequence x_1, x_2 is not regular in A, but we can explicitly find a regular element $x = v_1 + v_2 + v_3$ at the origin $A_{(0,0,0)}$. When passing to the quotient we get:

$$\begin{aligned} A/(x) &= k[v_1, v_2, v_3]/(v_1v_2, v_2v_3, v_1 + v_2 + v_3) \\ &= k[v_1, v_3]/(v_1(v_1 + v_3), v_3(v_1 + v_3)), \end{aligned}$$

The element $v_1 + v_3$ is killed by v_1 and v_3 and therefore by the maximal ideal $m = m_{(v_1,v_3)}$ of the local ring $(A/(x))_m$. This maximal ideal $m = \operatorname{Ann}(v_1 + v_3)$ and $H^0_m((A/(x))_m) \neq 0$, therefore $\operatorname{depth}_m(A/(x)) = 0$ and $\operatorname{depth}_{(0,0,0)}(A) = 1$ while $\dim(A_{(0,0,0)}) = 2$.

Example 2.7. *The ring $A = k[v_1, v_2, v_3, v_4, v_5]/(v_1v_2, v_2v_3)$ is analogous to the previous example, the dimension $\dim(A_m) = 4$ and the depth $\operatorname{depth}_m(A_m) = 3$ for the ideal $m = (v_1, v_2, v_3, v_4, v_5)$.*

Example 2.8. *The ring* $A = k[v_1, v_2, v_3, v_4, v_5]/(v_1v_2v_3, v_3v_4v_5)$ *is not Cohen-Macaulay. The irreducible components of the associated algebraic variety are, the four-dimensional component* $\Sigma_1 = V(v_3)$ *and the three dimensional component* $\Sigma_2 = V(v_1v_2, v_4v_5)$. *The dimension of the local rings are:*

(1) *Take* m *maximal ideal with* $(v_3) \subset m$ *but* (v_1v_2, v_4v_5) *not contained in* m*, we will have* $\dim(A_m) = \text{depth}_m(A_m) = 4$.
(2) *For* m *maximal ideal with* $(v_1v_2, v_4v_5) \subset m$ *but* (v_3) *not contained in* m*, we will have* $\dim(A_m) = \text{depth}_m(A_m) = 3$.
(3) *For a maximal idel* m *with* $(v_1v_2, v_4v_5) \cap (v_3) \subset m$*, the dimension* $\dim(A_m) = 4$ *while* $\text{depth}_m(A_m) \leq 3$.

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